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# *Different methods for obtaining the polar form of a Lorentz matrix M under the form :*

$$M = \begin{bmatrix} 1 + \frac{\gamma^2}{(1+\gamma)}\beta_1^2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_1 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_2^2 & \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_1 & \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_2 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_3^2 \end{bmatrix} .\Omega$$

*Summary : We present three different methods to obtain the polar form of a Lorentz matrix. We provide references for two additional methods.*

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*If the matrix* $M = \left( m^j_{\ i} \right)_{0 \le i \le n,\ 0 \le j \le n}$ *we note* ${}^tM = \left( m^i_{\ j} \right)_{0 \le i \le n,\ 0 \le j \le n}$.

*A Lorentz matrix* $M$ *is defined as a square matrix satisfying* ${}^tMGM = G$, *where* $G = \left( g^i_j \right)$ *with* $g^i_j = 0$ *si* $i \neq j$, $g^1_1 = 1$, $g^i_i = -1$ *for* $i > 1$.
*If* $M$ *is a* $4$ *th* $-$ *order matrix, it represents the transformation matrix from a spacetime basis of an observer* $O$ *to another spacetime basis associated with another observer* $O'$, *moving relative to* $O$ *with a uniform rectilinear motion of velocity* $\vec{V}$. *We aim to identify the terms of this matrix* $M$ *as a function of* $\vec{V}$ *or* $\vec{\beta} = \frac{\vec{V}}{c}$, $c$ *the lightspeed.*

## *First method :*

*The first method is a consequence of the theorems found in:*
*J-M. Souriau. "Calcul Linéaire" *PUF 1964 *p.378.*

***Theorem 1 :*** .

*Let* $X \in \mathbb{R}^n$ *such that* ${}^tXX = 1$ .
*Let* $N$ *be the matrix defined by:*

$0_{n-1}$ *being the zero matrix of* $M_{n-1}(\mathbb{R})$, $N = \begin{bmatrix} 0 & {}^tX \\ X & 0_{n-1} \end{bmatrix}$.

*Then,* $\forall \alpha \in \mathbb{R}$:

$$M = exp(\alpha N) = \begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \\ sh(\alpha)X & \left( Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X\,{}^tX \right) \end{bmatrix}$$

*is a Lorentz matrix of determinant equal to* $1$.

***Proof :***

*We have* $N^2 = \begin{bmatrix} 1 & 0 \\ 0^{n-1} & X\,{}^tX \end{bmatrix}$ *by noting* $0^{n-1}$ *as the zero column of order* $n-1$,
*since* ${}^tXX = 1$ *and then we have* $N^3 = N$.
*By noting that for any square matrix* $A$ *and* $\forall \alpha \in \mathbb{R}$,

$$exp(A) = \sum_{n=0}^{+\infty} \frac{A^n}{n!} \quad , \quad sh(\alpha) = \sum_{n=0}^{+\infty} \frac{\alpha^{2n+1}}{(2n+1)!} \quad , \quad ch(\alpha) = \sum_{n=0}^{+\infty} \frac{\alpha^{2n}}{(2n)!}$$

*therefore*

$$exp(\alpha N) = Id_{\mathbb{R}^4} + \alpha N + \frac{\alpha^2 N^2}{2!} + \frac{\alpha^3 N}{3!} + \frac{\alpha^4 N^2}{4!} + \ldots\ldots +$$
$$+ \frac{\alpha^{2p-1} N}{(2p-1)!} + \frac{\alpha^{2p} N^2}{(2p)!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$= Id_{\mathbb{R}^n} + sh(\alpha)N + (ch(\alpha) - 1)N^2$ .

*Now we can verify if M is indeed a Lorentz transformation :* ${}^tMGM = G$ *?*
*It is worth noting that* $M$ *is symmetric:* ${}^tM = M$ .

$$MG = \begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \\ sh(\alpha)X & \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X{}^tX\right) \end{bmatrix} \begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ sh(\alpha)X & -\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X{}^tX\right) \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ sh(\alpha)X & -\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X{}^tX\right) \end{bmatrix} \Big[ \begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \end{bmatrix}, \begin{bmatrix} (sh(\alpha))X & \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X{}^tX\right) \end{bmatrix} \Big]$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad : \quad \textit{Let's calculate } a, b, c, \textit{ and } d.$$

$a = ch^2(\alpha) - sh^2(\alpha)\,{}^tXX = 1$ , *because* ${}^tXX = 1$ .

$b = ch(\alpha)\ sh(\alpha)\,{}^tX - sh(\alpha)\,{}^tX\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X{}^tX\right)$
$= ch(\alpha)\ sh(\alpha)\,{}^tX - sh(\alpha)\,{}^tX - sh(\alpha)(ch(\alpha) - 1)\,{}^tXX{}^tX$
$= ch(\alpha)\ sh(\alpha)\,{}^tX - sh(\alpha)\,{}^tX - sh(\alpha)ch(\alpha)\,{}^tX + sh(\alpha){}^tX = 0$ ,

$c = ch(\alpha)\ sh(\alpha)X - \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X{}^tX\right)sh(\alpha)X$
$= ch(\alpha)\ sh(\alpha)X - sh(\alpha)X - sh(\alpha)(ch(\alpha) - 1)X = 0$ ,

$d = sh^2(\alpha)X{}^tX - \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X{}^tX\right)^2$ *but :*

$$\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X{}^tX\right)\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X{}^tX\right)$$

$= Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X{}^tX + (ch(\alpha) - 1)X{}^tX + (ch(\alpha) - 1)^2X{}^tXX{}^tX,$
*(It can be observed that as* ${}^tXX = 1$ *:* $X({}^tXX){}^tX = X$ *(using matrix multiplication associativity),*
*and* $(ch(\alpha) - 1)^2 = ch^2(\alpha) - 2ch(\alpha) + 1$)
$= Id_{\mathbb{R}^{n-1}} + (2(ch(\alpha) - 1) + ch^2(\alpha) - 2ch(\alpha) + 1)X{}^tX$

$= Id_{\mathbb{R}^{n-1}} + \left(ch^2(\alpha) - 1\right)X^tX = Id_{\mathbb{R}^{n-1}} + sh^2(\alpha)X^tX$

*hence :*

$sh^2(\alpha)X^tX - \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X^tX\right)^2 = -Id_{\mathbb{R}^{n-1}}$ .

*Consequently* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^{n-1}} \end{bmatrix}$

*and* $M$ *is a Lorentz matrix .*

***Another proof*** *:*

$$exp(\alpha N)G = \begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \\ sh(\alpha)X & \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X^tX\right) \end{bmatrix} \begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ sh(\alpha)X & -\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X^tX\right) \end{bmatrix} \text{ and}$$

$$Gexp(-\alpha N) = \begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^{n-1}} \end{bmatrix} \begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ -sh(\alpha)X & \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X^tX\right) \end{bmatrix}$$

$$= \begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ sh(\alpha)X & -\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X^tX\right) \end{bmatrix}$$

*hence* $exp(\alpha N)G = Gexp(-\alpha N) \Rightarrow exp(\alpha N)Gexp(\alpha N) = G$

*as* $M = exp(\alpha N)$ *is symmetric :* ${}^tMGM = G$ *and* $M$ *is a Lorentz matrix .*

***Lemma :***

*If* $M$ *is a Lorentz matrix and its first column* $K_1$ *has the form :* $K_1 = {}^t(\alpha, 0, ..., 0)$*, then* $M$ *has the form :*

$M = \begin{bmatrix} \varepsilon & {}^tQ \\ Q & \Omega \end{bmatrix}$ où $\varepsilon = \pm 1$ , ${}^tQ = (0, 0, 0)$, ${}^t\Omega\Omega = Id_{\mathbb{R}^{n-1}}$ $(1)$.

*Conversely, any matrix of this form* $(1)$ *is a Lorentz matrix.*

***Proof :***

*Let* $M = \begin{bmatrix} \alpha & {}^tL \\ 0 & C \end{bmatrix}$ *with* $C \in \mathbb{M}_3(\mathbb{R})$ , ${}^tL = (l_1, l_2, l_3)$ .

*Since* $({}^tMG)M = G$*, we have :*

$$G = \begin{bmatrix} \alpha & 0 \\ L & -{}^tC \end{bmatrix} \begin{bmatrix} \alpha & {}^tL \\ 0 & C \end{bmatrix} = \begin{bmatrix} \alpha^2 & \alpha\,{}^tL \\ \alpha L & L\,{}^tL - {}^tCC \end{bmatrix}.$$

*By identification, we obtain* $\alpha = \pm 1$ , $L = 0$, ${}^tCC = Id_{\mathbb{R}^{n-1}}$.

*If* $M$ *is of the form* $(1)$, *an immediate calculation shows that* ${}^tMGM = G$.

***Theorem 2 :***

*Any Lorentz matrix* $M$ *can be written in the form :*

$$M = exp(\alpha N) \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} \text{ where } \alpha \in \mathbb{R},\ N = \begin{bmatrix} 0 & {}^tX \\ X & O \end{bmatrix},\ X \in \mathbb{R}^{n-1},\ \varepsilon = \pm 1,$$

*such as :* ${}^tXX = 1$ , ${}^t\Omega\Omega = Id_{\mathbb{R}^{n-1}}$ .

***Proof :***

*Let* $M$ *be a Lorentz matrix* $M = (m[i, j])$
*and* $M[1] = (m[i, 1])$ *be the first column of* $M$ *then:*

$${}^tM_1GM_1 = m_{1,1}^2 - m_{1,2}^2 - \ldots - m_{1,n}^2 = G_{1,1} = 1$$

*We can set* ${}^tM_1 = (\beta, Y)$ *avec* $\beta = m_{1,1}$ *and* ${}^tY = (m_{1,2}, \ldots, m_{1,n})$. *Thus, we have* $\beta^2 - {}^tYY = 1$, *so* $|\beta| \geq 1$.

*We can set* $\beta = \varepsilon \cdot ch(\alpha)$, *where* $\varepsilon = \pm 1$ *and* $\alpha \in \mathbb{R}$.

*Additionally,* ${}^tYY = \beta^2 - 1 = sh^2(\alpha)$ .

*If* $\alpha \neq 0$, *let's set* $X = \dfrac{Y}{\varepsilon \cdot sh(\alpha)}$ , *then we have* ${}^tXX = 1$ .

*If* $\alpha = 0$, *we have* ${}^tM_1 = (\varepsilon, 0, 0, 0)$ .

*In all cases,* ${}^tM_1 = \varepsilon(ch(\alpha), sh(\alpha)X)$ *with* ${}^tXX = 1$.

*According to* ***lemma*** , $M[1]$ *is the first column of* $\varepsilon \cdot exp(\alpha N)$ *where* $N = \begin{bmatrix} 0 & {}^tX \\ X & O \end{bmatrix}$.

*We notice that* $exp(\alpha N) \cdot M$ *is a Lorentz matrix since* $M$ *and* $exp(-alpha N)$ *are.*
*Let's calculate the first column* $K[1]$ *of this product :*

$$K[1] = \begin{bmatrix} ch(\alpha) & (sh(\alpha))\,{}^tX \\ (sh(\alpha))X & Id_{\mathbb{R}^3} + (ch(\alpha) - 1)X\,{}^tX \end{bmatrix} \cdot M[1]$$

*By noting that* ${}^tXX = 1$ , $ch^2(\alpha) - sh^2(\alpha) = 1$ ,*and*
$\left(Id_{\mathbb{R}^3} + (ch(\alpha) - 1)X\,{}^tX\right) \cdot X = ch(\alpha) \cdot X$ ,
*we obtain :* ${}^tK_1 = (\varepsilon, 0, .., 0)$.
*We can then apply the lemma, and the theorem is proven.*

***Unicity of the decomposition :***

*The decomposition of the previous theorem is unique.*

***Proof :*** *It follows from the classical results that follow :*

***Lemma*** *:*

*Any square symmetric matrix* $A$ *of order* $n \geq 1$*, with real coefficients, is diagonalizable.*

*Moreover the eigenspaces of A are pairwise orthogonal. Therefore, we can write* $A = PD\,{}^tP$*,*
*where* $P$ *is an orthogonal matrix and* $D$ *is a diagonal matrix.*

***Lemma :***

*Let* $M$ *be an invertible matrix with real coefficients*
*. Then there exists a unique pair of matrices* $O$ *(orthogonal)*
*and* $S$ *(symmetric positive definite) such that* $M = SO$*.*

*Proof : The proof of this lemma can be found at the following link : https ://en • wikipedia • org/wiki*
*/Polar_decomposition.*

*This lemma implies that the decomposition in the theorem is unique.*

*For more details, refer to the following link : https ://en • wikipedia • org/wiki/Matrix_exponential •*

*Now let's apply these lemmas :*

*It is sufficient to verify that* $exp(\alpha N)$ *is positive definite. Since* $\alpha N$ *is a real symmetric matrix,*
*there exists a real orthogonal matrix* $U$ *and a real diagonal matrix* $D = (d_{i,\,i})$ *such that* $\alpha N = {}^tUDU$*.*

*As* $exp(\alpha N) = exp({}^tUDU) = {}^tUexp(D)U$ *, the eigenvalues of* $exp(\alpha N)$ *are strictly positive,*
*and exp(alphaN) is positive definite. From the formula* $det(e^A) = e^{Tr(A)}$*, we deduce that* $det(exp(\alpha N)) = 1$*,*

*and* $det(M) = \varepsilon \cdot det(\Omega) = \pm 1$ **.**

*Application :*

*Let* $M$ *be a Lorentz matrix, we have the following equivalences :*

$M$ *is symmetric* $\Leftrightarrow \quad M = exp(\alpha N)$

$M$ *is orthogonal* $\Leftrightarrow M = \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix}$

*For the direct parts, it is sufficient to note that :*

$$M = exp(\alpha N) \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} \text{ and } M = M \cdot Id \text{ or } M = Id \cdot M.$$

***Consequence :***

*We can now specify the polar form. By taking the previous notations, we will show that :*

*If we define* $\vec{V}$ *as the velocity of the observer* $O'$ *relative to the observer* $O$*,*
$c$ *as the speed of light, and* $\vec{\beta} = \vec{V}/c$*. We have if* $\gamma = \dfrac{1}{\sqrt{1 - \vec{\beta}^2}}$ *:*

$$exp(\alpha N) = \begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}] \\ [\gamma\vec{\beta}] & C \end{bmatrix} \text{ with } [\gamma\vec{\beta}] \text{ the coordinates of } \gamma\vec{\beta} \text{ in the frame of the observer } O ,$$

$$\vec{\beta} = th(\alpha)\vec{X},\ N = \begin{bmatrix} 0 & {}^tX \\ X & \Theta_{n-1} \end{bmatrix},\ \gamma = ch(\alpha) ,$$

$$C = \begin{bmatrix} 1 + \frac{\gamma^2}{(1+\gamma)}\beta^2_1 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_1 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta^2_2 & \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_1 & \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_2 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta^2_3 \end{bmatrix}.$$

***Proof :***

*Let us consider* $W' = \begin{bmatrix} ct' \\ 0 \\ 0 \\ 0 \end{bmatrix}$ *as the coordinates of* $O'$ *in the frame of the observer* $O'$.

*Also, let* $W = MW' = exp(\alpha N) \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} W'$ *be the coordinates of* $O'$ *in the frame of the observer* $O$,

*since* $M$ *is the transformation matrix from the frame of the observer* $O$ *to the frame of observer* $O'$,

*which are also equal to* $\begin{bmatrix} ct \\ c\beta_1 \\ c\beta_2 \\ c\beta_3 \end{bmatrix}$. *We first observe that*

$\begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} W' = W'$ *thus* $W = exp(\alpha N) W'$. *We are led to :*

$$\begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \\ sh(\alpha)X & \left(Id_{\mathbb{R}^3} + (ch(\alpha)-1)X^tX\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t' = \begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} t \quad , \text{ and}$$

$$\begin{bmatrix} ch(\alpha) \\ sh(\alpha)X_1 \\ sh(\alpha)X_2 \\ sh(\alpha)X_3 \end{bmatrix} t' = \begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} t \quad \text{and} \quad t = ch(\alpha)t' \text{ and consequently :}$$

$$ch(\alpha)\begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} ch(\alpha) \\ sh(\alpha)X_1 \\ sh(\alpha)X_2 \\ sh(\alpha)X_3 \end{bmatrix} \text{ for } t' \neq 0 \Rightarrow ch(\alpha)\vec{\beta} = sh(\alpha)\vec{X} \ ,$$

tthus $\vec{\beta} = th(\alpha)\vec{X} \Rightarrow \vec{\beta}^2 = th^2(\alpha)$ since $\vec{X}^2 = 1$.

We put $\beta = \sqrt{\vec{\beta}^2}$ et $\gamma = \dfrac{1}{\sqrt{1-\vec{\beta}^2}}$ .

As $1 - th^2(\alpha) = \dfrac{1}{ch^2(\alpha)} \Rightarrow ch^2(\alpha) = \dfrac{1}{1-th^2(\alpha)} = \dfrac{1}{1-\vec{\beta}^2} = \gamma^2$,

as $ch(\alpha) \geq 1 \quad \gamma = ch(\alpha)$; as $sh^2(\alpha) = ch^2(\alpha) - 1 = \gamma^2 - 1$

$$= \frac{1}{1-\vec{\beta}^2} - 1 = \frac{\beta^2}{1-\beta^2} = \gamma^2\beta^2 .$$

For short we have $\gamma = ch(\alpha)$ , $\gamma^2\beta^2 = sh^2(\alpha)$ , $\beta^2 = th^2(\alpha)$ .
Furthermore :

$$\gamma^2\beta^2 = \gamma^2 - 1 = (\gamma+1)(\gamma-1) \Rightarrow \frac{\gamma^2\beta^2}{(1+\gamma)} = (\gamma-1) = ch(\alpha) - 1$$

$$\text{et } X_iX_j = \frac{(\beta_i\beta_j)}{th^2(\alpha)} = \frac{(\beta_i\beta_j)}{\beta^2} \quad \text{consequently}$$

$$(ch(\alpha)-1)X_iX_j = \frac{\gamma^2\beta^2}{(1+\gamma)}\frac{(\beta_i\beta_j)}{\beta^2} = \frac{\gamma^2}{(1+\gamma)}\beta_i\beta_j, \text{ that implies:}$$

$$Id_{\mathbb{R}^3} + (ch(\alpha) - 1)X^tX = \begin{bmatrix} 1 + \frac{\gamma^2}{(1+\gamma)}\beta_1^2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_1 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_2^2 & \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_1 & \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_2 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_3^2 \end{bmatrix} = C$$

*As* $sh(\alpha)X_i = \frac{sh(\alpha)\beta_i}{th(\alpha)} = ch(\alpha)\beta_i = \gamma\beta_i$ *we have finally :*

$$exp(\alpha N) = \begin{bmatrix} \gamma & {}^t[\vec{\gamma\beta}] \\ [\vec{\gamma\beta}] & C \end{bmatrix} \text{ with } [\vec{\gamma\beta}] \text{ the coordinates of } \vec{\gamma\beta}.$$

*We notice that the inverse of* $exp(\alpha N)$ *is* $exp(-\alpha N)$*, simply by replacing* $\vec{\beta}$ *with* $-\vec{\beta}$ *to obtain the inverse while noting that* $\gamma$ *and* $C$ *remain unchanged, since :* $exp(\alpha N)\cdot exp(-\alpha N) = exp(0) = Id$

*if* $\vec{\beta}$ *is parallel to the space coordinate of* $\vec{i}$ *then only the diagonal term of* $C$ *are nonzero : 1* $+ \frac{\gamma^2}{(1+\gamma)}\beta_1^2$*, 1, 1 and*

$$1 + \frac{\gamma^2}{(1+\gamma)}\beta_1^2 = \frac{(1+\gamma) + \gamma^2\vec{\beta}^2)}{(1+\gamma)} = \frac{1+\gamma+\gamma^2-1}{(1+\gamma)} = \gamma \text{ because}$$

$$\gamma^2 - 1 = \frac{1}{1-\vec{\beta}^2} - 1 = \frac{\vec{\beta}^2}{1-\vec{\beta}^2} = \gamma^2\vec{\beta}^2.$$

***Other forms of expression of matrix M :***

*We can also write after the above;*

$$M = \begin{bmatrix} \varepsilon\cdot\gamma & {}^t[\vec{\gamma\beta}]\cdot\Omega \\ \varepsilon\cdot[\vec{\gamma\beta}] & C\cdot\Omega \end{bmatrix} = exp\left( Argch(\gamma) \begin{bmatrix} 0 & \frac{{}^t[\vec{\gamma\beta}]}{\beta} \\ \frac{[\vec{\gamma\beta}]}{\beta} & O \end{bmatrix} \right) \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix}.$$

***Another form of the polar form :***

*Furthermore if* $\varepsilon = 1$ **:**

*Let's write :* $\Lambda(\vec{\beta}) = \begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}] \\ [\gamma\vec{\beta}] & C([\vec{\beta}]) \end{bmatrix}$ *and* $\tilde{\Omega} = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$.

*We have*

$M = \Lambda(\vec{\beta}).\tilde{\Omega} = \tilde{\Omega}\cdot\left({}^t\tilde{\Omega}\cdot\Lambda(\vec{\beta}).\tilde{\Omega}\right) = \tilde{\Omega}\cdot\Lambda({}^t\Omega[\vec{\beta}])$ *because :*

$${}^t\tilde{\Omega}\cdot\Lambda(\vec{\beta}).\tilde{\Omega} = \begin{bmatrix} 1 & 0 \\ 0 & {}^t\Omega \end{bmatrix}\cdot\begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}] \\ [\gamma\vec{\beta}] & C([\vec{\beta}]) \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}] \\ {}^t\Omega\cdot[\gamma\vec{\beta}] & {}^t\Omega\cdot C \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} = \begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}]\cdot\Omega \\ {}^t\Omega\cdot[\gamma\vec{\beta}] & {}^t\Omega\cdot C\cdot\Omega \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\cdot{}^t({}^t\Omega[\vec{\beta}]) \\ \gamma\cdot{}^t\Omega\cdot[\vec{\beta}] & {}^t\Omega\cdot C\cdot\Omega \end{bmatrix}$$

*as* $C = Id + \frac{\gamma^2}{(1+\gamma)}[\vec{\beta}]\cdot{}^t[\vec{\beta}]$ *we have :*

$${}^t\Omega\cdot C\cdot\Omega = Id + \frac{\gamma^2}{(1+\gamma)}{}^t\Omega\cdot[\vec{\beta}]\cdot{}^t[\vec{\beta}]\cdot\Omega = Id + \frac{\gamma^2}{(1+\gamma)}({}^t\Omega[\vec{\beta}])\cdot{}^t({}^t\Omega[\vec{\beta}]).$$

## *Independence of Ω with respect to* $\vec{\beta}$ *when* $\varepsilon = +1$ *:*

*Now let's show that* $\Omega$ *does not depend on* $\vec{\beta}$. *We assume* $\varepsilon = +1\,(non-time\ inversion)$.
*Let* $M$ *be a Lorentz matrix of order* $4$*; it represents the transformation from a spacetime basis of an observer* $O$ *to another spacetime basis associated with another observer* $O'$*, who is in relative motion with respect to* $O$*, moving with a uniform velocity* $\vec{V}$.

*More precisely : Let* $\mathcal{B}(c\vec{\tau}, \vec{i}, \vec{j}, \vec{k})$ *be the basis associated with observer* $O$*, and* $\mathcal{B}'(c\vec{\tau}', \vec{i}', \vec{j}', \vec{k}')$ *be the basis associated with observer* $O'$.
$M = \mathcal{P}(\mathcal{B}(c\vec{\tau}, \vec{i}, \vec{j}, \vec{k}), \mathcal{B}'(c\vec{\tau}', \vec{i}', \vec{j}', \vec{k}'))$ *is the change − of − basis matrix .*

*A change − of − basis matrix* $\mathcal{P}(\mathcal{B}(c\vec{\tau}, \vec{i}, \vec{j}, \vec{k}), \mathcal{B}'(c\vec{\tau}', \vec{i}', \vec{j}', \vec{k}'))$ *is the representation of vectors* $(c\vec{\tau}', \vec{i}', \vec{j}', \vec{k}')$ *in the old basis* $\mathcal{B}(c\vec{\tau}, \vec{i}, \vec{j}, \vec{k})$

*For observer* $O$*, we consider the basis* $\mathcal{B}_1(c\vec{\tau}, \vec{i}_1, \vec{j}_1, \vec{k}_1)$ *such that the change − of − basis matrix* $A = \mathcal{P}(\mathcal{B}(c\vec{\tau}, \vec{i}, \vec{j}, \vec{k}), \mathcal{B}_1(c\vec{\tau}, \vec{i}_1, \vec{j}_1, \vec{k}_1))$ *is an orthogonal matrix of the form*

$A = \begin{bmatrix} 1 & 0 \\ 0 & R_1 \end{bmatrix}$ *where* $R_1$*, an orthogonal operator, transforms the spatial components of* $\vec{i}$ *to those of* $\vec{i}_1$ *such that they are parallel and have the same direction as* $\vec{\beta}$.

*Forthe observer* $O'$, *we consider the basis* $\mathcal{B}'_1\left(c\vec{\tau}', \vec{i}'_1, \vec{j}'_1, \vec{k}'_1\right)$ *such that the tchange* $-$ *of* $-$ *basis matrix*

$^tB = \mathcal{P}\left(\mathcal{B}'\left(c\vec{\tau}', \vec{i}', \vec{j}', \vec{k}'\right), \mathcal{B}'_1\left(c\vec{\tau}', \vec{i}'_1, \vec{j}'_1, \vec{k}'_1\right)\right)$ *is an orthogonal matrix of the form*

$^tB = \begin{bmatrix} 1 & 0 \\ 0 & R_2 \end{bmatrix}$ *où* $R_2$, *an orthogonal operator, transforms the spatial components of* $\vec{i}'$ *to those of* $\vec{i}'_1$ *such that it is parallel and has the same direction as* $\vec{\beta}$, *and* $\vec{j}'_1$ *and* $\vec{k}'_1$ *are parallel and have the same direction as* $\vec{j}_1$, *and* $\vec{k}_1$ *respectively,* (*rotation around the first axis*).

*Let's calculate the operator* $N$ *that allows us to transform the basis* $\mathcal{B}\left(c\vec{\tau}, \vec{i}_1, \vec{j}_1, \vec{k}_1\right)$ *to the basis* $\mathcal{B}'\left(c\vec{\tau}', \vec{i}'_1, \vec{j}'_1, \vec{k}'_1\right)$ :

$N = \mathcal{P}\left(\mathcal{B}\left(c\vec{\tau}, \vec{i}_1, \vec{j}_1, \vec{k}_1\right), \mathcal{B}'\left(c\vec{\tau}', \vec{i}'_1, \vec{j}'_1, \vec{k}'_1\right)\right)$.

*Since* $\vec{\beta}$ *is perpendicular to the spatial components of* $\vec{j}_1$ *and* $\vec{k}_1$, *we have :*

$$N = \begin{bmatrix} \gamma & \gamma\text{-}\beta & 0 & 0 \\ \gamma\text{-}\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega_{1,1} & \omega_{1,2} & \omega_{1,3} \\ 0 & \omega_{2,1} & \omega_{2,2} & \omega_{2,3} \\ 0 & \omega_{3,1} & \omega_{3,2} & \omega_{3,3} \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta\omega_{1,1} & \gamma\beta\omega_{1,2} & \gamma\beta\omega_{1,3} \\ \gamma\beta & \gamma\omega_{1,1} & \gamma\omega_{1,2} & \gamma\omega_{1,3} \\ 0 & \omega_{2,1} & \omega_{2,2} & \omega_{2,3} \\ 0 & \omega_{3,1} & \omega_{3,2} & \omega_{3,3} \end{bmatrix}.$$

*As the spatial components of the new spatial vectors are parallel and in the same direction as those of the old one , we necessarily have :*

$\omega_{2,1} = \omega_{3,1} = \omega_{3,2} = \omega_{1,2} = \omega_{2,3} = \omega_{1,3} = 0$ *et* $\omega_{1,1} > 0$ *then :*

$$N = \begin{bmatrix} \gamma & \gamma\beta\omega_{1,1} & 0 & 0 \\ \gamma\beta & \gamma\omega_{1,1} & 0 & 0 \\ 0 & 0 & \omega_{2,2} & 0 \\ 0 & 0 & 0 & \omega_{3,3} \end{bmatrix},$$

*as* $\gamma^2 - \gamma^2\beta^2 = 1$ *and* $^tN \bullet G \cdot N = G$ *imply that* $\gamma^2\beta^2\omega^2_{1,1} - \gamma^2\omega_{1,1} = -1$ ,

*and we have* $\omega_{1,1} = \omega_{2,2} = \omega_{3,3} = 1$ *et* $N = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

*Now we have :*

$M = \mathcal{P}\left(\mathcal{B}\left(c\vec{\tau}, \vec{i}, \vec{j}, \vec{k}\right), \mathcal{B}'\left(c\vec{\tau}', \vec{i}', \vec{j}', \vec{k}'\right)\right)$

$= \mathcal{P}\left(\mathcal{B}\left(c\vec{\tau}, \vec{i}, \vec{j}, \vec{k}\right), \mathcal{B}\left(c\vec{\tau}, \vec{i}_1, \vec{j}_1, \vec{k}_1\right)\right) \mathcal{P}\left(\mathcal{B}\left(c\vec{\tau}, \vec{i}_1, \vec{j}_1, \vec{k}_1\right), \mathcal{B}'\left(c\vec{\tau}', \vec{i}'_1, \vec{j}'_1, \vec{k}'_1\right)\right) \mathcal{P}\left(\mathcal{B}'\left(c\vec{\tau}', \vec{i}'_1, \vec{j}'_1, \vec{k}'_1\right), \mathcal{B}'\left(c\vec{\tau}', \vec{i}', \vec{j}', \vec{k}'\right)\right)$

$$= A \bullet N \bullet B = \left(A \bullet N \bullet {}^{t}A\right) \bullet (A \bullet B) = \begin{bmatrix} \gamma & {}^{t}[\gamma\vec{\beta}] \\ [\gamma\vec{\beta}] & C \end{bmatrix} . \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} .$$

*By the unicity of the decomposition, we have* $\begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} = A \bullet B$

*and* $\begin{bmatrix} \gamma & {}^{t}[\gamma\vec{\beta}] \\ [\gamma\vec{\beta}] & C \end{bmatrix} = A. \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . {}^{t}A .$

$\Omega$ *is therefore independent of* $\vec{\beta}$ *since* $A$ *and* $B$ *are independent of* $\vec{\beta}$

*As* $\lim_{\vec{\beta} \to 0} \left( \begin{bmatrix} \gamma & {}^{t}[\gamma\vec{\beta}] \\ [\gamma\vec{\beta}] & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$ ,

$\begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$ *is the Lorentz matrix* $M$ *for* $\vec{\beta} = \vec{0}$ .

## *Transformation of Galileo :*

*This also allows us to show that the Galileo transformation is the limit as* $c \to +\infty$
*of the Lorentz transformation . We notice that if we write the Lorentz matrix in the form :*

$$M = \begin{bmatrix} \gamma & {}^{t}[\gamma\vec{\beta}] \\ [\gamma\vec{\beta}] & Id_{\mathbb{R}^3} + \frac{\gamma^2}{(1+\gamma)} [\vec{\beta}] \cdot {}^{t}[\vec{\beta}] I \end{bmatrix} . \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & {}^{t}[\gamma\vec{\beta}]\Omega \\ [\gamma\vec{\beta}] & \left( Id_{\mathbb{R}^3} + \frac{\gamma^2}{(1+\gamma)} [\vec{\beta}] \cdot {}^{t}[\vec{\beta}] \right) \Omega \end{bmatrix} .$$

*On the other hand :*

$$MY' = \begin{bmatrix} \gamma & {}^{t}[\gamma\vec{\beta}]\Omega \\ [\gamma\vec{\beta}] & \left( Id_{\mathbb{R}^3} + \frac{\gamma^2}{(1+\gamma)} [\vec{\beta}] \cdot {}^{t}[\vec{\beta}] \right) \Omega \end{bmatrix} . \begin{bmatrix} ct' \\ X' \end{bmatrix}$$

$$= \begin{bmatrix} c\gamma & {}^{t}[\gamma\vec{\beta}]\Omega \\ c[\gamma\vec{\beta}] & \left( Id_{\mathbb{R}^3} + \frac{\gamma^2}{(1+\gamma)}[\vec{\beta}] \cdot {}^{t}[\vec{\beta}] \right)\Omega \end{bmatrix} \cdot \begin{bmatrix} t' \\ X' \end{bmatrix} = \begin{bmatrix} ct \\ X \end{bmatrix}$$

*and then :*

$$\begin{bmatrix} \gamma & {}^{t}[\gamma\vec{\beta}]\frac{\Omega}{c} \\ c[\gamma\vec{\beta}] & \left( Id_{\mathbb{R}^3} + \frac{\gamma^2}{(1+\gamma)}[\vec{\beta}] \cdot {}^{t}[\vec{\beta}] \right)\Omega \end{bmatrix} \cdot \begin{bmatrix} t' \\ X' \end{bmatrix} = \begin{bmatrix} t \\ X \end{bmatrix}$$

*since* $[\vec{\beta}] = \left[\frac{\vec{V}}{c}\right]$ *and* $\gamma = \frac{1}{1 - \left(\frac{\vec{V}}{c}\right)^2}$ *hence :*

$$\begin{bmatrix} \gamma & {}^{t}\left[\gamma\frac{\vec{V}}{c}\right]\frac{\Omega}{c} \\ [\gamma\vec{V}] & \left( Id_{\mathbb{R}^3} + \frac{\gamma^2}{(1+\gamma)}\left[\frac{\vec{V}}{c}\right] \cdot {}^{t}\left[\frac{\vec{V}}{c}\right] \right)\Omega \end{bmatrix} \cdot \begin{bmatrix} t' \\ X' \end{bmatrix} = \begin{bmatrix} t \\ X \end{bmatrix} \text{ then}$$

*For a given velocity* $\vec{V}$, $\Omega$ *is independent of* $c$,

*if* $c \to +\infty$ *, we have* $\begin{bmatrix} 1 & 0 \\ [\vec{V}] & \Omega \end{bmatrix} \cdot \begin{bmatrix} t' \\ X' \end{bmatrix} = \begin{bmatrix} t \\ X \end{bmatrix}$,

*that is the Galileo transformation .*

## *Calculation of eigenvalues;*

*We can determine the eigenvalues directly :*

$$As\ \Lambda = \begin{bmatrix} 1 + \frac{\gamma^2}{(1+\gamma)}\beta_1^2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_1 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_2^2 & \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_1 & \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_2 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_3^2 \end{bmatrix} = A. \begin{bmatrix} \gamma & \gamma\text{-}\beta & 0 & 0 \\ \gamma\text{-}\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

$${}^{T}A ,$$

*where* $A$ *est orthogonal :* $A.{}^{T}A = Id$ .

*we note that* $\det(\Lambda - \lambda Id) = \det\left( A.\begin{bmatrix} \gamma & \gamma\text{-}\beta & 0 & 0 \\ \gamma\text{-}\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .{}^{T}A - \lambda .A.{}^{T}A \right)$

$$= \det(A)\det\left( \begin{bmatrix} \gamma & \gamma\text{-}\beta & 0 & 0 \\ \gamma\text{-}\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \lambda .Id \right) \det({}^{T}A).$$

*and*

$$\begin{bmatrix} \gamma & \gamma\text{-}\beta & 0 & 0 \\ \gamma\text{-}\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ +\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \gamma\text{-}(1+\beta) & 0 & 0 & 0 \\ 0 & \gamma\text{-}(1-\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \frac{\sqrt{2}}{2} & +\frac{\sqrt{2}}{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

*In the general case, it is shown that every Lorentz matrix admits at least one real eigenvalue :*
$(cf : E \bullet GOURGOULON : Relativité\ restreinte\ EDP\ Sciences)$.
*As it is of order* **4**, *it has* **2** *eigenvalues. This is the maximum number for the general case :*
*consider the case where M is orthogonal to* $\Omega$, *a rotation matrix around the* **Ox** *axis.*

## *Another expression of* $\Omega$ :

*We can directly calculate* $\Omega$ *because*
*if* $M = \Lambda(\vec{\beta}).\tilde{\Omega} \Leftrightarrow \Omega = \Lambda(-\vec{\beta}) \bullet M$.
*It is possible to have a simpler calculation :*
*We have* $\left(\vec{\beta}\ {}^{t}\vec{\beta}\right)\left(\vec{\beta}\ {}^{t}\vec{\beta}\right) = \vec{\beta}\left({}^{t}\vec{\beta}\vec{\beta}\right){}^{t}\vec{\beta} = \vec{\beta}^{2}\left(\vec{\beta}\ {}^{t}\vec{\beta}\right)$ *and* $\gamma^{2}\vec{\beta}^{2} = \gamma^{2} - 1$,
*that imply that :*

$$\left( Id_{\mathbb{R}^3} + \frac{\left[\gamma\vec{\beta}\right]^{t}\left[\gamma\vec{\beta}\right]}{(1+\gamma)} \right) . \left( Id_{\mathbb{R}^3} - \gamma\frac{\vec{\beta}\ {}^{t}\vec{\beta}}{(1+\gamma)} \right)$$

$$= Id_{\mathbb{R}^3} - \gamma\frac{\vec{\beta}\ {}^{t}\vec{\beta}}{(1+\gamma)} + \gamma^{2}\frac{\vec{\beta}\ {}^{t}\vec{\beta}}{(1+\gamma)} - \gamma^{3}\frac{\vec{\beta}^{2}\left(\vec{\beta}\ {}^{t}\vec{\beta}\right)}{(1+\gamma)^{2}}$$

$$= Id_{\mathbb{R}^3} + \frac{\vec{\beta}\ {}^{t}\vec{\beta}}{(1+\gamma)^{2}}\left( -\gamma(1+\gamma) + \gamma^{2}(1+\gamma) - \gamma\left(\gamma^{2}-1\right) \right) = Id_{\mathbb{R}^3}.$$

Then $C^{-1} = \left( Id_{\mathbb{R}^3} + \frac{[\gamma\vec{\beta}]\,{}^t[\gamma\vec{\beta}]}{(1+\gamma)} \right)^{-1} = Id_{\mathbb{R}^3} - \gamma\frac{\vec{\beta}\,{}^t\vec{\beta}}{(1+\gamma)}$

As $M = \begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}] \\ [\gamma\vec{\beta}] & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} = \begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}]\Omega \\ [\gamma\vec{\beta}] & C\Omega \end{bmatrix}$,

Now if we write $M = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix}$.

We have $\Omega = \left( Id_{\mathbb{R}^3} - \gamma\frac{\vec{\beta}\,{}^t\vec{\beta}}{(1+\gamma)} \right) \begin{bmatrix} m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,2} & m_{3,3} & m_{3,4} \\ m_{3,2} & m_{4,3} & m_{4,4} \end{bmatrix}$.

***A final remark :***

*We can note :*

*M symmetric* $\Leftrightarrow$ *the spatial components of the spatial vectors of* $\mathcal{B}(c\vec{\tau}, \vec{i}, \vec{j}, \vec{k})$ *and* $\mathcal{B}'(c\vec{\tau}', \vec{i}', \vec{j}', \vec{k}')$ *are parallel .*

*Now we propose two other ways to calculate the terms of* $exp(\alpha N)$ *:*

***Second method :***

*The second method starts with a demonstration by J. Parizet in "La géométrie de la relativité restreinte (Ellipses) p.89."*

*Let's M a Lorentz matrix :* $G = {}^tMGM$ *that implies* $GM^{-1} = {}^tMG$ *and* $M^{-1} = G{}^tMG$.

*We can write M as* $\begin{bmatrix} \gamma & {}^tU \\ V & A \end{bmatrix}$ *où* $\gamma \in \mathbb{R}$, *U et V 2 vecteurs de* $\mathbb{R}^3$ *et A une matrice 3 x3 .*

*from* $M^{-1} = G{}^tMG$ *we have* $M^{-1} = \begin{bmatrix} \gamma & -{}^tV \\ -U & {}^tA \end{bmatrix}$ ***and then :***

$M \cdot M^{-1} = \begin{bmatrix} \gamma^2 - {}^tV \cdot V & \gamma \cdot {}^tU - {}^tV \cdot A \\ -\gamma U + {}^tA.V & {}^tA \cdot A - U \cdot {}^tU \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Id_3 \end{bmatrix}$ *that implies :*

$\gamma^2 - {}^tV \cdot V = 1$, $-\gamma U + {}^tA.V = \gamma \cdot {}^tU - {}^tV \cdot A = 0$, ${}^tA \cdot A - U \cdot {}^tU = Id_3$.

We note $\gamma^2 \geq 1$ since $\gamma^2 - 1 = {}^tV \cdot V \geq 0$, ${}^tA.V = \gamma.U$ and ${}^tV \cdot A = \gamma \cdot {}^tU$.

We have $V = 0 \Leftrightarrow \gamma^2 = 1$, $V = 0 \Leftrightarrow U = 0$, $V = 0 \Leftrightarrow {}^tA \cdot A = Id_3$.

Now **if** $V = 0$ then $M = \begin{bmatrix} \pm 1 & 0 \\ 0 & A \end{bmatrix}$ with ${}^tA \cdot A = Id_3$.

Consider the case $V \neq 0$ and then $\gamma^2 > 1$.

We then consider $\Omega = A - \dfrac{V \cdot {}^tU}{1+\gamma}$.

Let' calculate ${}^t\Omega\ \Omega = \left({}^tA - \dfrac{U.{}^tV}{1+\gamma}\right)\left(A - \dfrac{V \cdot {}^tU}{1+\gamma}\right)$

$= {}^tA \cdot A - \dfrac{{}^tAV\,{}^tU}{1+\gamma} - \dfrac{U\,{}^tVA}{1+\gamma} + \dfrac{U\,{}^tVV\,{}^tU}{(1+\gamma)^2}$,

$= Id_3 + U \cdot {}^tU - \dfrac{2.\gamma U.{}^tU}{1+\gamma} + \dfrac{\left(\gamma^2 - 1\right)U \cdot {}^tU}{\left(\gamma^2 - 1\right)}$

$= Id_3 + U \cdot {}^tU\left(1 - \dfrac{2.\gamma}{1+\gamma} + \dfrac{\left(\gamma^2 - 1\right)}{(1+\gamma)^2}\right)$

$= Id_3 + U \cdot {}^tU(1 + \gamma - 2.\gamma + \gamma - 1) = Id_3$.

We conclude that $\Omega$ is an orthogonal matrix.

Now we can consider :

${}^tV.\ \Omega = {}^tV \cdot \left(A - \dfrac{V \cdot {}^tU}{1+\gamma}\right) = {}^tV \cdot A - \dfrac{{}^tV \cdot V \cdot {}^tU}{1+\gamma} = \gamma \cdot {}^tU - \dfrac{\left(\gamma^2 - 1\right) \cdot {}^tU}{1+\gamma} = {}^tU$.

Then $\Omega \cdot U = \Omega\,{}^t\Omega \cdot V = V \cdot$ Because $\Omega = A - \dfrac{V \cdot {}^tU}{1+\gamma}$

$$M = \begin{bmatrix} \gamma & {}^tU \\ V & A \end{bmatrix} = \begin{bmatrix} \gamma & {}^tU \\ V & \Omega + \dfrac{V \cdot {}^tU}{1+\gamma} \end{bmatrix} = \begin{bmatrix} \gamma & {}^tU \cdot {}^tO \\ V & Id + \dfrac{V \cdot {}^tU \cdot {}^t\Omega}{1+\gamma} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$$ and then :

$$M = \begin{bmatrix} \gamma & {}^tV \\ V & Id + \dfrac{V\,{}^tV}{1+\gamma} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}.$$

If we consider $O$ et $O'$, $O'$ with a constant speed $\vec{v}$ with respect to $O$ :

$O'$ has the coordinate $\begin{bmatrix} c \cdot t' \\ 0_{\mathbb{R}^3} \end{bmatrix}$ in the basis associated with $O'$ and

$\begin{bmatrix} c \cdot t \\ [\vec{v}] \cdot t \end{bmatrix}$ in the basis associated with $O$.

*Then :* $\begin{bmatrix} c\cdot t \\ \left[\vec{V}\right]\cdot t \end{bmatrix} = \begin{bmatrix} \gamma & {}^{t}V \\ V & Id + \dfrac{V\,{}^{t}V}{1+\gamma} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} c\cdot t' \\ 0_{\mathbb{R}^3} \end{bmatrix}$.

*we deduce from it :* $t = \gamma\cdot t'$ *and* $\left[\vec{V}\right]\cdot t = c\cdot t'\cdot V$ *then* $V = \dfrac{\gamma\left[\vec{V}\right]}{c} = \gamma\left[\vec{\beta}\right]$

*Finally :*

$$M = \begin{bmatrix} \gamma & \gamma\cdot {}^{t}\left[\vec{\beta}\right] \\ \gamma\left[\vec{\beta}\right] & Id + \dfrac{\gamma^2\left[\vec{\beta}\right]{}^{t}\left[\vec{\beta}\right]}{1+\gamma} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}.$$

*We can also write* $M = \begin{bmatrix} -\gamma & -\gamma\left[\overleftrightarrow{\beta}\right] \\ -\gamma\left[\overleftrightarrow{\beta}\right] & -\left(Id + \dfrac{\gamma^2\left[\vec{\beta}\right]\left[\overleftrightarrow{\beta}\right]}{1+\gamma}\right) \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -\Omega \end{bmatrix}$ *and*

*In a general way* $M$ *can be written as :*

$$M = \begin{bmatrix} \gamma & \gamma\cdot {}^{t}\left[\vec{\beta}\right] \\ \gamma\left[\overleftrightarrow{\beta}\right] & Id + \dfrac{\gamma^2\left[\vec{\beta}\right]{}^{t}\left[\vec{\beta}\right]}{1+\gamma} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} \text{ with } \varepsilon = \pm 1,\ \gamma \geq 1,\ \gamma = \frac{1}{\sqrt{1 - {}^{t}\vec{\beta}.\vec{\beta}}}$$

*with* $\Omega$ *an orthonal matrix .*

## *Third method :*

*We start from: Special relativity by Ulrich E. Schröder, page 92, World Scientific, modifying his demonstration.*

*(The book can be consulted at : https ://archive.org/details/specialrelativit0000schr/page/92/mode/2 up"?view=theater).*

*Following Schröder, let us consider a symmetric Lorentz matrix* $M$ *of the form :*

$$M = A.\begin{bmatrix} \gamma & \gamma\cdot\beta & 0 & 0 \\ \gamma\cdot\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot {}^{T}A \quad \text{with } \beta = \left\|\vec{\beta}\right\|,\ A = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix},\ \Omega \text{ an orthogonal matrix } 3\,x3\,.$$

$M$ *is indeed a Lorentz matrix, since it can be expressed as the product of* $3$ *Lorentz matrices.*

*Since* $\Omega$ *is a* $3\,x3$ *rotation matrix,*

*it can be decomposed into a product of* $3$ *elementary rotations (Euler angles) :*

*see, for example, https ://en•wikipedia•org/wiki/Rotation_formalisms_in_three_dimensions•*

*If we define :* $Rx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$, $Ry = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & 0 & \sin(\varphi) \\ 0 & 0 & 1 & 0 \\ 0 & -\sin(\varphi) & 0 & \cos(\varphi) \end{bmatrix}$, $Rz = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\psi) & -\sin(\psi) & 0 \\ 0 & \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

*We can write* $A = Rz \bullet Ry \bullet Rx$ .

*This order of composition is chosen to simplify the calculations.*

*It is observed that due to rotational invariance around the the* $x-$ *axis.*

$$Rx \cdot \begin{bmatrix} \gamma & \gamma\cdot\beta & 0 & 0 \\ \gamma\cdot\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \bullet {}^{t}Rx = \begin{bmatrix} \gamma & \gamma\cdot\beta & 0 & 0 \\ \gamma\cdot\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

*After calculation* $(with\ Maple)$ *:*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\psi)\cos(\varphi) & -\sin(\psi)\cos(\theta) + \cos(\psi)\sin(\varphi)\sin(\theta) & \sin(\psi)\sin(\theta) + \cos(\psi)\sin(\varphi)\cos(\theta) \\ 0 & \sin(\psi)\cos(\varphi) & \cos(\psi)\cos(\theta) + \sin(\psi)\sin(\varphi)\sin(\theta) & -\cos(\psi)\sin(\theta) + \sin(\psi)\sin(\varphi)\cos(\theta) \\ 0 & -\sin(\varphi) & \cos(\varphi)\sin(\theta) & \cos(\varphi)\cos(\theta) \end{bmatrix}.$$

*We have since the* $\theta$ *variable is eliminated :*

$$M = \begin{bmatrix} \gamma & \cos(\psi)\cos(\varphi)\gamma\beta & \sin(\psi)\cos(\varphi)\gamma\beta & -\sin(\varphi)\gamma\beta \\ \cos(\psi)\cos(\varphi)\gamma\beta & 1+\cos(\psi)^2(-1+\gamma)\cos(\varphi)^2 & \sin(\psi)\cos(\psi)\cos(\varphi)^2(-1+\gamma) & -\cos(\varphi)\cos(\psi)\sin(\varphi)(-1+\gamma) \\ \sin(\psi)\cos(\varphi)\gamma\beta & \sin(\psi)\cos(\psi)\cos(\varphi)^2(-1+\gamma) & 1+\left((1-\gamma)\cos(\psi)^2-1+\gamma\right)\cos(\varphi)^2 & -\cos(\varphi)\sin(\psi)\sin(\varphi)(-1+\gamma) \\ -\sin(\varphi)\gamma\beta & -\cos(\varphi)\cos(\psi)\sin(\varphi)(-1+\gamma) & -\cos(\varphi)\sin(\psi)\sin(\varphi)(-1+\gamma) & (1-\gamma)\cos(\varphi)^2+\gamma \end{bmatrix}.$$

*One can still directly evaluate the first column of a general Lorentz matrix with* $\vec{\beta}$ *by considering the coordinates of the observer* $O'$ *in the two bases associated with* $O$ *and* $O'$*:*

*If we write* $M = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix}$ *then* $M \cdot \begin{bmatrix} ct' \\ 0 \\ 0 \\ 0 \end{bmatrix}_{o'} = \begin{bmatrix} ct \\ c\cdot\beta_1\cdot t \\ c\cdot\beta_2\cdot t \\ c\cdot\beta_3\cdot t \end{bmatrix}_{o}$ *hence :*

*if we write* $\gamma = m_{1,1}$ *then* $\gamma\cdot t' = t$ *moreover* $\gamma > 0 \cdot (Principle\ of\ causality)$ .

*From this, we deduce that the first column of* $M$ *is written as* $\begin{bmatrix} \gamma \\ \gamma\cdot\beta_1 \\ \gamma\cdot\beta_2 \\ \gamma\cdot\beta_3 \end{bmatrix}$.

*By identification :* $\cos(\psi)\cos(\varphi)=\frac{\beta_1}{\beta}$ , $\sin(\psi)\cos(\varphi)=\frac{\beta_2}{\beta}$ , $\sin(\varphi)=-\frac{\beta_3}{\beta}$ .

*We recall that* $\gamma^2-\beta^2=\gamma^2-1$ *since* $1=\gamma^2(1-\beta^2)$ *since* ${}^tMGM=G$.

*Calculation of the* $m_{i,j}$ *:*

$$m_{2,2}=1+\cos(\psi)^2(-1+\gamma)\cos(\varphi)^2=1+\cos(\psi)^2\cos(\varphi)^2\frac{\gamma^2-\beta^2}{1+\gamma}$$

$$=1+\left(\frac{\beta_1}{\beta}\right)^2\frac{\gamma^2-\beta^2}{1+\gamma}\qquad=1+\frac{\gamma^2}{1+\gamma}\beta_1^2 .$$

$$m_{3,3}=1+\left((1-\gamma)\cos(\psi)^2-1+\gamma\right)\cos(\varphi)^2=1+(1-\gamma)\left(\cos(\psi)^2\cos(\varphi)^2-\cos(\varphi)^2\right)$$

$$=1+(1-\gamma)\left(\cos(\psi)^2\cos(\varphi)^2-1+\sin(\varphi)^2\right)=1+(1-\gamma)\left(\frac{\beta_1^2-\beta^2+\beta_3^2}{\beta^2}\right)$$

$$=1+\frac{\gamma^2}{1+\gamma}\cdot\beta_2^2 .$$

$$m_{4,4}=(1-\gamma)\cos(\varphi)^2+\gamma=1+(\gamma-1)(1-\cos(\varphi)^2)=1+\frac{\gamma^2-\beta^2}{1+\gamma}\sin(\varphi)^2=1+\frac{\gamma^2}{1+\gamma}\beta_3^2 .$$

$$m_{3,2}=\sin(\psi)\cos(\psi)\cos(\varphi)^2(-1+\gamma)=\sin(\psi)\cos(\varphi)\cos(\psi)\cos(\varphi)(-1+\gamma)=\frac{\gamma^2-\beta^2}{1+\gamma}\frac{\beta_2}{\beta}\frac{\beta_1}{\beta}$$

$$=\frac{\gamma^2}{1+\gamma}\beta_1\beta_2 .$$

$$m_{4,3}=-\cos(\varphi)\sin(\psi)\sin(\varphi)(-1+\gamma)=\frac{\gamma^2}{1+\gamma}\beta_2\beta_3 .$$

$$m_{4,2}=-\cos(\varphi)\cos(\psi)\sin(\varphi)(-1+\gamma)=\frac{\gamma^2}{1+\gamma}\beta_1\beta_3 .$$

*We complete by symmetry .*

***Other methods :***

*See also*

***E • GOURGOULON*** *: Relativité restreinte EDP Sciences p.203 •*
*a geometric method based on the existence of an eingenvalue ,*
*and for a heuristic approach :*

***C • Semay B • Silvestre — Brac*** *Relativité Restreinte Dunod p.120 .*

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